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COMMENT

A note on the eigenvalues of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ for spin-1 in a constant magnetic field

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Abstract. Exploiting the intimate connection of the problem of a charged spin-1 particle in a homogeneous magnetic field to that of a harmonic oscillator, we demonstrate explicitly that the eigenvalues of the matrix operator $\boldsymbol{S}\cdot\boldsymbol{\pi}$ for spin-1 and for a constant magnetic field are governed by a Hermitian matrix defined on the space of the particle number n and spin and are thus constrained to be real for any intensity of the external magnetic field H, thereby contradicting a recent affirmation of Weaver that for $n=0$, complex eigenvalues are present for sufficiently intense magnetic fields. We also point out the errors that have crept into Weaver's analysis which have led to complex eigenvalues and as well to an excessive number of eigenvalues compared with that demanded by the spin degrees of freedom.

In a recent paper Weaver (1978) has observed that the eigenvalue spectrum of the matrix operator $\boldsymbol{S}\cdot\boldsymbol{\pi}$ where S_i $(i=1,2,3)$ are the spin-1 matrices, $\boldsymbol{\pi}=\boldsymbol{p}-e\boldsymbol{A}=-\mathrm{i}\nabla-e\boldsymbol{A}$, e the charge and $(A_0,\boldsymbol{A})=(0,\frac{1}{2}H(-y,x,0))$, the 4-vector potential for a constant magnetic field H in the z direction, also includes complex values depending on the intensity of H. Weaver's analysis has also led to an excessive number of eigenvalues, in fact three times the number demanded by the number of spin polarisations for spin-1. As the nature (purely real or not) of the eigenvalue spectrum of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ has an important bearing on the nature of energy eigenvalues of spin-1 Hamiltonians (see, for instance, Weaver's treatment (Weaver 1976) of the Sakata–Taketani (Taketani and Sakata 1940) Hamiltonian for spin-1 with a specific anomalous coupling with a constant magnetic field) and there exists a general connection between the possible occurrence of complex energy eigenvalues and the problem of acausality of propagation for spin-1 (see for instance, Krase *et al* 1971, Goldman and Tsai 1971, Prabhakaran and Seetharaman 1973, Mathews 1974), it is pertinent to reinvestigate as to the exact nature of the eigenvalue spectrum of $\boldsymbol{S}\cdot\boldsymbol{\pi}$ for spin-1. This is our objective in this comment.

Following the essential lines of Mathews (1974) converting the problem into the corresponding one of the harmonic oscillator, we demonstrate explicitly in § 2 that such a conversion leads to a Hermitian matrix for the corresponding eigenvalue problem in the space of the particle number n and spin for all $n=0,1,2,\ldots$ and for any intensity of the external magnetic field, thereby negating Weaver's conclusions regarding the emergence of complex eigenvalues for $n=0$. Our analysis also leads to the correct number of eigenvalues demanded by the three spin orientations for spin-1, thus rectifying an undue number of eigenvalues stemming from Weaver's analysis. In

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§ 3 we present a comparison of our results with that of Weaver and point out the errors involved in Weaver's analysis of the problem.

We shall now proceed to prove our assertions. Starting with the $\boldsymbol{\pi}$ components

$$\pi_1 = p_1 - eA_1 = -\mathrm{i}\,\partial/\partial x - \tfrac{1}{2}eyH, \qquad \pi_2 = p_2 - eA_2 = -\mathrm{i}\,\partial/\partial y + \tfrac{1}{2}exH,$$
$$\pi_3 = p_3 = -\mathrm{i}\,\partial/\partial z, \tag{1a, b, c}$$

it follows (Mathews 1974) that the operators defined by

$$a = (2eH)^{-1/2}\pi_+, \qquad a^\dagger = (2eH)^{-1/2}\pi_-, \qquad \pi_\pm = \pi_1 \pm \mathrm{i}\pi_2, \tag{2a, b}$$

together with the number operator

$$N = a^\dagger a \tag{3}$$

satisfy the algebra

$$[a, a^\dagger]_- = 1, \qquad [N, a]_- = -a, \qquad [N, a^\dagger]_- = a^\dagger \tag{4}$$

equivalent to that of a simple harmonic oscillator and that

$$[\pi_3, a]_- = [\pi_3, a^\dagger]_- = 0. \tag{5}$$

By virtue of (5), π_3 can be replaced by its eigenvalue

$$\pi_3 \to p_3 = (2eH)^{1/2}a_3, \qquad a_3 \text{ any real number.} \tag{6}$$

It readily follows that

$$\pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2 = 2eH(N + \tfrac{1}{2} + a_3^2). \tag{7}$$

Now, starting with the eigenvalue problem

$$\boldsymbol{S} \cdot \boldsymbol{\pi}\psi = \lambda\psi \tag{8}$$

for spin-1, the same can be written in the form

$$(2eH)^{1/2}[\tfrac{1}{2}(S_+a^\dagger + S_-a) + S_3a_3]\psi = \lambda\psi \tag{9}$$

where $S_\pm = S_1 \pm \mathrm{i}S_2$.

Defining (Mathews 1974) a complete set of orthonormal states $|n, \alpha_i\rangle = |n\rangle \otimes |\alpha_i\rangle$, ($n = 0, 1, 2, \ldots, i = 1, 2, 3$; $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = -1$), in the space of the number operator N and spin, the following properties of $|n, \alpha_i\rangle$ follow readily. (For a fuller meaning of the number eigenstates $|n\rangle$, see Mathews 1974.)

$$S_3|n, \alpha_i\rangle = \alpha_i|n, \alpha_i\rangle, \qquad i = 1, 2, 3, \qquad \alpha_1 = 1, \alpha_2 = 0, \alpha_3 = -1, \tag{10a}$$

$$S_+|n, \alpha_1\rangle = 0, \qquad S_+|n, \alpha_2\rangle = \sqrt{2}|n, \alpha_1\rangle, \qquad S_+|n, \alpha_3\rangle = \sqrt{2}|n, \alpha_2\rangle, \tag{10b}$$

$$S_-|n, \alpha_1\rangle = \sqrt{2}|n, \alpha_2\rangle, \qquad S_-|n, \alpha_2\rangle = \sqrt{2}|n, \alpha_3\rangle, \qquad S_-|n, \alpha_3\rangle = 0; \tag{10c}$$

$$\langle n_1, \alpha_i|n_2, \alpha_j\rangle = \delta_{n_1n_2}\delta_{\alpha_i\alpha_j}, \qquad n_1, n_2 = 0, 1, 2 \ldots, \qquad \alpha_i, \alpha_j = 1, 0, -1. \tag{11}$$

Expanding ψ of (9) in terms of the basis $|n, \alpha_i\rangle$, we have the expansion

$$\psi = \sum_{n=0}^{\infty} c_{n\alpha_1}|n, \alpha_1\rangle + \sum_{n=0}^{\infty} c_{n\alpha_2}|n, \alpha_2\rangle + \sum_{n=0}^{\infty} c_{n\alpha_3}|n, \alpha_3\rangle \tag{12}$$

where $c_{n\alpha_i}$ ($i = 1, 2, 3$) are the expansion coefficients. Note that $c_{n\alpha_i}$ for $n < 0$ do not find a place in (12) and hence $c_{n\alpha_i} \equiv 0$ for $n < 0$. This observation is crucial for the ensuing analysis of the eigenvalues λ.

Substituting (12) in (9) and making use of the properties (10*a*)-(10*c*) of $|n, \alpha_i\rangle$, we obtain, after a simplification, that

$$\begin{aligned}(2eH)^{1/2}\Bigg\{\sum_n\left[a_3\alpha_1 c_{n\alpha_1}+\left(\frac{n}{2}\right)^{1/2}c_{(n-1)\alpha_2}\right]|n,\alpha_1\rangle \\ +\sum_n\left[a_3\alpha_2 c_{n\alpha_2}+\left(\frac{n}{2}\right)^{1/2}c_{(n-1)\alpha_3}+\left(\frac{n+1}{2}\right)^{1/2}c_{(n+1)\alpha_1}\right]|n,\alpha_2\rangle \\ +\sum_n\left[a_3\alpha_3 c_{n\alpha_3}+\left(\frac{n+1}{2}\right)^{1/2}c_{(n+1)\alpha_2}\right]|n,\alpha_3\rangle\Bigg\} \\ =\lambda\left(\sum_n c_{n\alpha_1}|n,\alpha_1\rangle+\sum_n c_{n\alpha_2}|n,\alpha_2\rangle+\sum_n c_{n\alpha_3}|n,\alpha_3\rangle\right).\end{aligned} \tag{13}$$

Now, by virtue of the orthonormality property, (11), of $|n, \alpha_i\rangle$, equation (13) results in a set of equations for the coefficients $c_{n\alpha_1}$, $c_{(n-1)\alpha_2}$ and $c_{(n-2)\alpha_3}$, which can be written in the form of the following eigenvalue equation for a 3×3 matrix A for $n=2, 3, \ldots$:

$$Ac=(2eH)^{1/2}\begin{pmatrix} a_3 & \left(\frac{n}{2}\right)^{1/2} & 0 \\ \left(\frac{n}{2}\right)^{1/2} & 0 & \left(\frac{n-1}{2}\right)^{1/2} \\ 0 & \left(\frac{n-1}{2}\right)^{1/2} & -a_3\end{pmatrix}\begin{pmatrix} c_{n\alpha_1} \\ c_{(n-1)\alpha_2} \\ c_{(n-2)\alpha_3}\end{pmatrix}=\lambda\begin{pmatrix} c_{n\alpha_1} \\ c_{(n-1)\alpha_2} \\ c_{(n-2)\alpha_3}\end{pmatrix}, \tag{14}$$

$$n=2, 3, \ldots,$$

where c is the column vector $(c_{n\alpha_1}, c_{(n-1)\alpha_2}, c_{(n-2)\alpha_3})$. As A is Hermitian the eigenvalues λ are real for any intensity of the external magnetic field H and can be obtained from the roots of the cubic characteristic equation

$$|A-\lambda I(3\times3)|=0 \tag{15a}$$

i.e.

$$\lambda^3-[2eH(n-\tfrac{1}{2})+p_3^2]\lambda-eHp_3=0, \qquad n=2, 3, \ldots, \tag{15b}$$

thus affirming that only three real eigenvalues of λ occur for each $n=2, 3, \ldots$ in accordance with the three possible polarisations for spin-1.

However the cases $n=0$ and $n=1$ occur as special ones and the Hermitian matrix A is effectively uni-dimensional and bi-dimensional respectively in view of the fact that $c_{n\alpha_i}\equiv0$ for $n<0$. For $n=0$, we obtain specifically that

$$Ac=((2eH)^{1/2}a_3)(c_{0\alpha_1})=(c_{0\alpha_1}), \qquad \text{for } n=0, \tag{16}$$

with the only real eigenvalue being given by

$$\lambda=(2eH)^{1/2}a=p_3. \tag{17}$$

For $n=1$, we obtain the eigenvalue equation

$$Ac=(2eH)^{1/2}\begin{pmatrix} a_3 & 2^{-1/2} \\ 2^{-1/2} & 0\end{pmatrix}\begin{pmatrix} c_{1\alpha_1} \\ c_{0\alpha_2}\end{pmatrix}=\lambda\begin{pmatrix} c_{1\alpha_1} \\ c_{0\alpha_2}\end{pmatrix}, \qquad \text{for } n=1. \tag{18}$$

Because of Hermiticity of the 2×2 matrix A, the eigenvalues are real for any intensity of the external magnetic field and are given by the roots of the quadratic equation

$$\lambda^2 - \lambda p_3 - eH = 0 \tag{19}$$

i.e.

$$\lambda = \tfrac{1}{2}[p_3 \pm (p_3^2 + 4eH)^{1/2}]. \tag{20}$$

Our results above testify our affirmation that only real eigenvalues occur for $\boldsymbol{S}\cdot\boldsymbol{\pi}$ for spin-1 for all $n \geqslant 0$ and for arbitrary intensity of the external magnetic field.

Let us now proceed to make a comparison of our results with those of Weaver and to identify the mistakes in Weaver's analysis.

For his analysis of the problem Weaver (1978) starts with the Kemmer algebra

$$S_iS_jS_k + S_kS_jS_i = S_i\delta_{jk} + S_k\delta_{ji}, \qquad i, j, k = 1, 2, 3, \tag{21}$$

satisfied by the spin-1 matrices, contracts both sides of (21) with the product $\pi_i\pi_j\pi_k$ multiplied from the right, utilises the familiar angular momentum commutation relations

$$[S_i, S_j]_- = \mathrm{i}\varepsilon_{ijk}S_k, \qquad i, j, k = 1, 2, 3, \tag{22}$$

and the commutation relations of the $\boldsymbol{\pi}$ components,

$$[\pi_i, \pi_j]_- = \mathrm{i}e\varepsilon_{ijk}H_k, \qquad \boldsymbol{H} = (0, 0, H), \tag{23}$$

so as to deduce that

$$(\boldsymbol{S}\cdot\boldsymbol{\pi})^3 - (\pi^2 - 2eS_3H)(\boldsymbol{S}\cdot\boldsymbol{\pi}) - eHp_3 = 0. \tag{24}$$

Weaver then combines (24) with the eigenvalue equation (8) for $\boldsymbol{S}\cdot\boldsymbol{\pi}$ and employs the substitution of the operator $\pi^2 - 2eHS_3$ by its eigenvalue

$$\begin{aligned} &\pi^2 - 2eHS_3 \to 2eH(n_1 + \tfrac{1}{2} - m_s) + p_3^2 = 2eH[(n_1 - m_s + 1) - \tfrac{1}{2}] + p_3^2, \\ &n_1 = 0, 1, 2, \ldots, \qquad m_s = \pm 1, 0, \end{aligned} \tag{25}$$

so as to deduce that λ satisfies the cubic equation

$$\begin{aligned} &\lambda^3 - \{2eH[(n_1 - m_s + 1) - \tfrac{1}{2}] + p_3^2\}\lambda - eHp_3 = 0, \\ &n_1 = 0, 1, 2, \ldots, \qquad m_s = \pm 1, 0. \end{aligned} \tag{26}$$

We observe that equation (26) with n_1 and m_s taking independently values 0, 1, 2, ... and $\pm 1, 0$ respectively is incorrect. The basic error lies in the fact that though $\boldsymbol{S}\cdot\boldsymbol{\pi}$ commutes with $\pi^2 - 2eHS_3$, it does not commute separately with π^2 and $2eHS_3$ (though π^2 and $2eHS_3$ commute mutually) as is evident from the following commutation relations

$$\begin{aligned} [\boldsymbol{S}\cdot\boldsymbol{\pi}, \pi^2]_- &= (2eH)^{3/2}[\tfrac{1}{2}(S_+a^\dagger + S_-a) + S_3a_3, N + \tfrac{1}{2} + a_3^2]_- \\ &= -\tfrac{1}{2}(2eH)^{3/2}(S_+a^\dagger - S_-a) \neq 0, \end{aligned} \tag{27a}$$

$$\begin{aligned} [\boldsymbol{S}\cdot\boldsymbol{\pi}, 2eS_3H]_- &= (2eH)^{3/2}[\tfrac{1}{2}(S_+a^\dagger + S_-a) + S_3a_3, S_3]_- \\ &= -\tfrac{1}{2}(2eH)^{3/2}(S_+a^\dagger - S_-a) \neq 0, \end{aligned} \tag{27b}$$

$$[\boldsymbol{S}\cdot\boldsymbol{\pi}, \pi^2 - 2eS_3H]_- = 0, \tag{27c}$$

$$[\pi^2, S_3H]_- = (2eH)[N + \tfrac{1}{2} + a_3^2, S_3H]_- = 0. \tag{27d}$$

Hence $\boldsymbol{S} \cdot \boldsymbol{\pi}$, π^2 and $2eHS_3$ cannot be diagonalised simultaneously. However Weaver has assumed such a diagonalisation for his deduction of (26). Because of this error, equation (26) suggests that there is a cubic equation for independent values of $n_1 = 0, 1, 2, \ldots$ and $m_s = \pm 1, 0$ leading thus to the same roots repeated three times more than the number admissible by the number of polarisations for spin-1. Note that in contrast to this defect of Weaver's equation (26), our equation (15*b*) indeed gives the correct number of roots, namely three, for each $n \geqslant 2$.

In fact comparing our equation (15*b*) for $n \geqslant 2$ with that of Weaver (26), it is clear that Weaver's $(n_1 - m_s + 1)$ should be equal to our 'n' i.e.

$$n = n_1 - m_s + 1 \tag{28a}$$

and moreover the factor $n_1 - m_s + 1$ in Weaver's equation (26) should be modified to assume values $2, 3, \ldots$ etc in a conjoint manner, i.e.

$$n = n_1 - m_s + 1 = 2, 3, \ldots \tag{28b}$$

in order to avoid untenable multiplicity of repeated roots.

For $n = 0$, which corresponds to Weaver's $(n_1 = 0,\ m_s = 1)$, Weaver's procedure still leads to a cubic equation (26) namely

$$\lambda^3 - (p_3^2 - eH)\lambda - eHp_3 = 0 \tag{29}$$

which can be factorised into

$$(\lambda - p_3)(\lambda^2 + \lambda p_3 + eH) = 0 \tag{30}$$

while our direct procedure leads to the linear equation (17) which corresponds to the first linear factor part of (30). We now make the important observation that the complex values for λ in Weaver's analysis originate exactly from the extraneous roots

$$\lambda = \tfrac{1}{2}[-p_3 \pm (p_3^2 - 4eH)^{1/2}] \tag{31}$$

coming from the extra quadratic factor part in (30) when $4eH > p_3^2$. Since the case $n = 0$ occurs in our analysis as a special one, the matrix A of (14) being uni-dimensional, we draw the evident conclusion that Weaver's procedure of contraction of the spin-1 algebra (21) with the product $\pi_i \pi_j \pi_k$ to obtain uniformly a cubic equation (24) has unwantonly destroyed the minimality of the characteristic equation for the matrix A of (16) in the corresponding eigenvalue problem in the space of the number operator and spin and thereby has led to the emergence of complex eigenvalues for λ if $4eH > p_3^2$. However our analysis shows that such complex roots are not present at all.

For the case $n = 1$, which corresponds to Weaver's $(n_1 = 0,\ m_s = 0)$ and $(n_1 = 1,\ m_s = 1)$, Weaver's procedure has led to a cubic equation (26) for each of these two sets, namely

$$\lambda^3 - (eH + p_3^2)\lambda - eHp_3 = 0 \tag{32}$$

which can indeed be factorised into

$$(\lambda + p_3)(\lambda^2 - p_3\lambda - eH) = 0, \tag{33}$$

while our direct procedure leads to the quadratic equation (19) which corresponds to the second quadratic factor part of (33). Since the case $n = 1$ also occurs in our analysis as a special one, the matrix A of (18) being bi-dimensional, we draw a similar conclusion that Weaver's procedure of contraction to obtain the cubic equation (24) has again destroyed the minimality of the characteristic equation for the corresponding matrix

A of (18) in the corresponding eigenvalue problem in the space of the number operator and spin and thereby has led to an extra root $\lambda = -p_3$ which however is not present as is clear from our analysis.

Our work here establishing that the eigenvalue spectrum of the matrix operator $\boldsymbol{S} \cdot \boldsymbol{\pi}$ for spin-1 is purely real for any intensity of the external constant magnetic field has served to remove the false impression in the recent literature (Weaver 1976, 1978) that this eigenvalue spectrum also includes complex values for sufficiently intense magnetic fields. We have also pointed out the errors that have crept into Weaver's analysis (Weaver 1978) leading to such complex eigenvalues and as well to an excessive number of eigenvalues compared with that demanded by the three spin polarisations for spin-1. Our procedure which is essentially an adaptation to the problem considered here of the one used earlier by Mathews (1974) for determining the energy spectrum of a spin-1 relativistic particle with charge in a constant magnetic field, is vastly simpler than the procedure of Weaver (1978) contracting the spin-1 algebra with suitable products of π_i and moreover leads to correct results as have been analysed in this comment. Our procedure is more amenable for extension to the case of arbitrary spin than that of Weaver as even for spin-$\frac{3}{2}$, for example, the algebra used by Weaver (1978), which is equivalent (Jayaraman 1981) to the algebra of Bhabha and Madhava Rao (Corson 1953) is sufficiently complicated not to mention the complexity of an algebra for arbitrary spin! In fact the results of our current calculations easily extending the results of this comment for spin-1 to the case of arbitrary spin will be published separately.

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